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**GRAPHS OF THE REAL AND IMAGINARY PARTS  
OF THE EFFECTIVE COLLISION FREQUENCY  
FOR WEAKLY IONIZED AIR**

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8625-6001-RU-000

Contract No. AF19(604)-8844

Scientific Report #1

*January 1962*

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AIR FORCE CAMBRIDGE RESEARCH LABORATORIES  
OFFICE OF AEROSPACE RESEARCH  
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BEDFORD, MASSACHUSETTS

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ABSTRACT

Graphs of the real and imaginary parts of the effective collision frequency for air are presented. The propagation constants for an E.M. wave propagating in ionized air with a static magnetic field are presented in terms of these effective collision frequencies. A numerical example illustrating the use of these graphs is worked out.

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# I. Introduction

In a previous paper<sup>1</sup> it was shown that the use of the Langevin equation to describe the conductivity of a weakly ionized gas is incorrect unless the dissipative term  $g$  is interpreted as complex. Limiting forms of both the real and imaginary parts of  $g$  were given under either low or high pressure conditions. Analytical expressions for conductivity valid for all pressures were presented for several gases including air. Phelps<sup>4</sup> has also presented the correct analytical expressions for the conductivity of air.

The Langevin equation reads as follows:

$$(1) \quad \frac{dv}{dt} + gv = \frac{q}{m} E_0 e^{i\omega t}$$

where  $v$  is the average drift velocity of electrons in the gas,  $g$  is the dissipative term, often erroneously called the "collision frequency" and  $q = -e$ ;  $m$  and  $E_0$  have their usual meanings. Equation (1) and its steady state solution:

$$(2) \quad v = \frac{1}{i\omega + g} \left( \frac{q}{m} \right) E_0 e^{i\omega t}$$

are the starting points for many discussions on the conductivity and propagation constants of weakly ionized gases.<sup>1,2,3</sup>

Our purpose in this paper is to present in graphical form the values of  $g = g_r + ig_i$  for intermediate values of  $\frac{\omega}{\nu}$ <sup>5</sup> where  $\nu$  is the average collision frequency<sup>1,4</sup>. We call  $g$  the "effective collision frequency" and  $g_r$  and  $g_i$  the real and imaginary parts respectively. We see that knowledge of their values enables us immediately to correct the previous analysis in (for example) references 2 and 3; for: The usual starting point is

$$(2') \quad v = \frac{1}{i\omega + \nu} \left(\frac{q}{m}\right) E_0 e^{i\omega t}$$

whereas the correct relationship is

$$(2'') \quad v = \frac{1}{i(\omega + g_i) + g_r} \left(\frac{q}{m}\right) E_0 e^{i\omega t}$$

Thus to correct the already available analyses whenever an expression for conductivity,  $\sigma(\omega, \nu)$ , occurs just replace  $\omega$  by  $\omega + g_i$  and  $\nu$  by  $g_r$  (i.e.,  $\sigma(\omega + g_i, g_r)$ ). The values of  $g_i$  and  $g_r$  are determined by the ratio  $\frac{\omega}{\nu}$ . We give some examples of the application of this method below drawing heavily on the results of reference 1.

## II. The Conductivity Tensor<sup>1</sup>

The conductivity tensor  $\|\sigma\|$  is given by

$$(4) \quad \|\sigma\| = \frac{\omega_p^2}{8\pi} \begin{vmatrix} L + R & i(R - L) & 0 \\ i(L + R) & L + R & 0 \\ 0 & 0 & 2P \end{vmatrix}$$

where

$$(5) \quad P = \frac{1}{g_r + i(\omega + g_r)}$$

$$(6) \quad L = \frac{1}{g_r^+ + i(\omega + \omega_b + g_1^+)}$$

$$(7) \quad R = \frac{1}{g_r^- + i(\omega - \omega_b + g_1^-)}$$

where  $\omega_b = \frac{q|H|}{mc} = \frac{eH}{mc}$  the cyclotron frequency and  $\omega_p^2 = \frac{4\pi n e^2}{m}$ .  
 $g_r^+$  and  $g_1^+$  are obtained by looking up their values in the graphs (Figs. 1 and 2) under the appropriate value of  $\frac{\omega + \omega_b}{\gamma}$ . Thus should  $|\omega| = |\omega_b|$ , then in (7),  $g_1^- = 2(\omega - \omega_b) = 0$  and  $g_r^- = \gamma$ ; whereas in (6)  $g_r^+$  could range anywhere from  $\gamma$  to  $\frac{5}{3}\gamma$  and  $g_1^+$  from 0 to  $2(\omega + \omega_b)$  depending on the magnitude of  $\frac{\omega + \omega_b}{\gamma}$ .

### III. The Propagation Constants

The propagation constants for an electromagnetic wave in a conducting gas in a magnetic field have been given already by Phelps<sup>6</sup> in terms of the conductivity.

In the language of effective collision frequencies, we

---

\* Alternatively  $\omega P = \omega B \left( \frac{3\omega}{2\gamma} \right) - i\omega D \left( \frac{3\omega}{2\gamma} \right)$ ,

$$(\omega + \omega_b)L = (\omega + \omega_b) \left[ B \left( \frac{3}{2} \frac{\omega + \omega_b}{\gamma} \right) - iD \left( \frac{3}{2} \frac{\omega + \omega_b}{\gamma} \right) \right]$$

$$(\omega - \omega_b)R = (\omega - \omega_b) \left[ B \left( \frac{3}{2} \frac{\omega - \omega_b}{\gamma} \right) - iD \left( \frac{3}{2} \frac{\omega - \omega_b}{\gamma} \right) \right]$$

and  $\omega B$  and  $\omega D$  are graphed. See reference 5.

present the following propagation constants (paraphrasing Phelps<sup>6</sup>).

a) Propagation parallel to the magnetic field

$$(8) \begin{Bmatrix} K_o \\ K_x \end{Bmatrix} = \frac{\omega}{c} \left\{ 1 - \frac{\omega_p^2}{\omega} \begin{Bmatrix} L \\ R \end{Bmatrix} \right\}^{\frac{1}{2}} = \frac{\omega}{c} \left\{ 1 - \frac{\left[ \omega_p^2 (\omega \pm \omega_b + g_1^{\pm}) + i \omega_p^2 g_r^{\pm} \right]}{\omega \left[ (g_r^{\pm})^2 + (\omega \pm \omega_b + g_1^{\pm})^2 \right]} \right\}^{\frac{1}{2}}$$

The wave associated with the + sign is the ordinary wave and is left circularly polarized about the direction of propagation. The minus sign is associated with the extraordinary wave which is right circularly polarized.

Writing  $K = \mu - ik$  where  $\mu$  is the real part of the refractive index and  $k$  is the absorption coefficient and assuming  $\frac{\omega_p^2}{\omega(\omega - \omega_b)} \ll 1$ .

We obtain

$$(9) \begin{Bmatrix} \mu_o \\ \mu_x \end{Bmatrix} = \frac{\omega}{c} \left\{ 1 - \frac{\omega_p^2}{2\omega} \frac{(\omega \pm \omega_b + g_1^{\pm})}{[(\omega \pm \omega_b + g_1^{\pm})^2 + (g_r^{\pm})^2]} \right\}$$

$$(10) \begin{Bmatrix} k_o \\ k_x \end{Bmatrix} = \frac{\omega_p^2}{2c} \frac{g_r^{\pm}}{[(\omega \pm \omega_b + g_1^{\pm})^2 + (g_r^{\pm})^2]}$$

b) Propagation perpendicular to the magnetic field

$$(11) K_o = \frac{\omega}{c} \left\{ 1 - \frac{i \omega_p^2}{\omega} P \right\}^{\frac{1}{2}} = \frac{\omega}{c} \left\{ 1 - \frac{\omega_p^2}{\omega} \frac{[\omega + g_1 + i g_r]}{(\omega + g_1)^2 + g_r^2} \right\}^{\frac{1}{2}}$$

$$(12) k_x = \frac{\omega}{c} \left\{ 1 + \frac{\omega_p^2 (L + R) + 2 \frac{\omega^4}{i \omega} RL}{2i \omega + \omega_p^2 (R + L)} \right\}^{\frac{1}{2}} *$$

\*The equivalent Equation (21) of Phelps appears to be in error.

When  $\frac{\omega_p^2}{\omega^2} \ll 1$  then

$$\mu_o = \frac{\omega}{c} \left[ 1 - \frac{\omega_p^2}{2\omega} \frac{(\omega + g_i)}{[(\omega + g_i)^2 + (g_r)^2]} \right] \quad (13)$$

$$k_o = \frac{\omega_p^2}{2c} \frac{g_r}{[(\omega + g_i)^2 + g_r^2]}$$

Also when  $\frac{\omega_p^2}{\omega(\omega - \omega_b)} \ll 1$

$$\mu_x = \frac{\omega}{c} \left[ 1 - \frac{\omega_p^2}{4\omega} \frac{(\omega + \omega_b + g_i^+)}{[(\omega + \omega_b + g_i^+)^2 + (g_r^+)^2]} - \frac{\omega_p^2}{4\omega} \frac{(\omega - \omega_b + g_i^-)}{[(\omega - \omega_b + g_i^-)^2 + (g_r^-)^2]} \right] \quad (14)$$

$$k_x = \frac{\omega_p^2}{4c} \left[ \frac{g_r^+}{(\omega + \omega_b + g_i^+)^2 + (g_r^+)^2} + \frac{g_r^-}{(\omega - \omega_b + g_i^-)^2 + (g_r^-)^2} \right]$$

#### IV. A Numerical Example

Let  $\omega = 2\nu = 2\omega_b$ , (i.e., suppose we are propagating a wave of angular frequency  $\omega$  in ionized air where according to Phelps' relationship  $[\nu = 1.04 \times 10^8 \text{ p sec}^{-1}]$  the collision frequency is half that of the angular frequency). Then find the absorption coefficient for the extraordinary mode of a wave propagating perpendicular to the magnetic field (i.e.,  $k_x$  of Equation 14) assuming

$$\frac{\omega_p^2}{\omega(\omega - \omega_b)} \ll 1 .$$

Since

$$\frac{\omega + \omega_b}{\nu} = \frac{\omega(1 + \frac{1}{2})}{\nu} = 2 \times \frac{3}{2} = 3 ,$$

thus

$$g_r^+ = 1.52\nu , g_i^+ = .07(\omega + \omega_b)$$

and

$$\frac{\omega - \omega_b}{\nu} = \frac{\omega(1 + \frac{1}{2})}{\nu} = 2 \times \frac{1}{2} = 1,$$

then

$$g_r^- = 1.31\nu$$

and

$$g_i^- = .23 (\omega - \omega_b) .$$

Then

$$\begin{aligned} k_x &= \frac{\omega_p^2}{4c} \left[ \frac{1.52\nu}{(1.07)^2(\omega + \omega_b)^2 + (1.52)^2\nu^2} + \frac{1.31\nu}{(1.23)^2(\omega - \omega_b)^2 + (1.31)^2\nu^2} \right] \\ &= \frac{\omega_p^2}{4c} \left[ \frac{1.52}{(1.07)^2 \times 9 + (1.52)^2} + \frac{1.31}{(1.23)^2 + (1.31)^2} \right] = .525 \frac{\omega_p^2}{4c\nu} \end{aligned}$$

The conventional formulation would have

$$k_x = \frac{\omega_p^2}{4c} \left[ \frac{\nu}{(\omega + \omega_b)^2 + \nu^2} + \frac{\nu}{(\omega - \omega_b)^2 + \nu^2} \right]$$

instead of (14), and substituting in the above values for  $\omega$ ,

$\nu$  and  $\omega_b$ , we get

$$k_x = \frac{\omega_p^2}{4c\nu} \left[ \frac{1}{9 + 1} + \frac{1}{1 + 1} \right] = \frac{\omega_p^2}{4c\nu} \quad [.6]$$

Thus using the incorrect formulation we incur an error of 14%.

# APPENDIX

## The Propagation Constant for Arbitrary Direction of Propagation

Sen and Wyller<sup>7</sup> have given the propagation constant for propagation in an arbitrary direction. We reproduce their formula in the language of the components of the conductivity tensor, equations (5), (6), (7).

$$A1) \quad k^2 = \frac{A + B \sin^2 \varphi \pm \sqrt{B^2 \sin^4 \varphi - C^2 \cos^2 \varphi}}{D + E \sin^2 \varphi}$$

Where  $\varphi$  is the angle between the magnetic field and the direction of propagation and

$$A = 2\epsilon_1 (\epsilon_1 + \epsilon_3)$$

$$B = \epsilon_3 (\epsilon_1 + \epsilon_3) + \epsilon_2^2$$

$$C = 2\epsilon_1 \epsilon_2, \quad D = 2\epsilon_1, \quad E = 2\epsilon_3$$

and

$$\epsilon_1 = 1 - \frac{i\omega p}{\omega} P$$

$$\epsilon_2 = \frac{1}{2} \frac{\omega p}{\omega} [R - L]$$

$$\epsilon_3 = \frac{i\omega p}{\omega} \left[ P - \frac{1}{2} (L + R) \right]$$

FOOTNOTES

1. P. Molmud, Phys. Rev., 114, 29 (1959).
2. S. K. Mitra, "The Upper Atmosphere", (The Asiatic Society, Calcutta, 1952), 2nd ed.
3. J. A. Ratcliffe, "The Magneto-Ionic Theory and Its Application of the Ionosphere" (Cambridge U. Press, Cambridge, England, 1959).
4. A. V. Phelps, J. Ap. Phys., 31, No. 10, p. 1723, (1960).  
These expressions are exactly equivalent to those the author presented in reference 1. Phelps parameter  $\frac{1}{\gamma}$  is my  $X = \frac{3\omega}{2\nu}$  and  $\nu$  is the average collision frequency. For air  $\nu$  is given by Phelps as:  $\nu = 1.04 \times 10^8 p \text{ sec}^{-1}$  where p is the pressure in mm. Hg.
5.  $g_r$  and  $g_i$  are computed from analytical expressions given in reference 1. See particularly Equations 4 and A3 and A4 in this reference where  $\omega B = \frac{5}{2} X C_{5/2}(X)$  and  $\omega D = X^2 C_{3/2}(X)$ . The latter two functions have been tabulated by Dingle, Arnt and Roy, Appl. Sci. Research 6B, 155, (1956).  
 $\omega B$  and  $\omega D$  may be found graphed in P. Molmud, "The Electrical Conductivity of Weakly Ionized Air" R. W., GM 47-78, 30 Sept. 1958, Phelps' <sup>4</sup> function  $A_2$  and  $B_2$  are equivalent respectively to  $\omega B$  and  $\omega D$  where  $\omega$  is either  $\omega$ ,  $\omega + \omega_b$  or  $\omega - \omega_b$ .
6. Phelps, Op. Cit., p. 1727, Eq. (16) thru (23). To translate his A's and B's to effective collision frequencies simply employ the following translations

$$A(\omega \pm \omega_b) \rightarrow \frac{g_r^{\pm} (\omega \pm \omega_b)}{(\omega \pm \omega_b + g_i^{\pm})^2 + (g_r^{\pm})^2}$$

$$B(\omega \pm \omega_b) \rightarrow \frac{(\omega \pm \omega_b + g_i^{\pm})(\omega \pm \omega_b)}{(\omega \pm \omega_b + g_i^{\pm})^2 + (g_r^{\pm})^2}$$

7. H. K. Sen and A. A. Wyller, Journal of Geophysical Research, Vol. 65, No. 12, Dec. 1960.



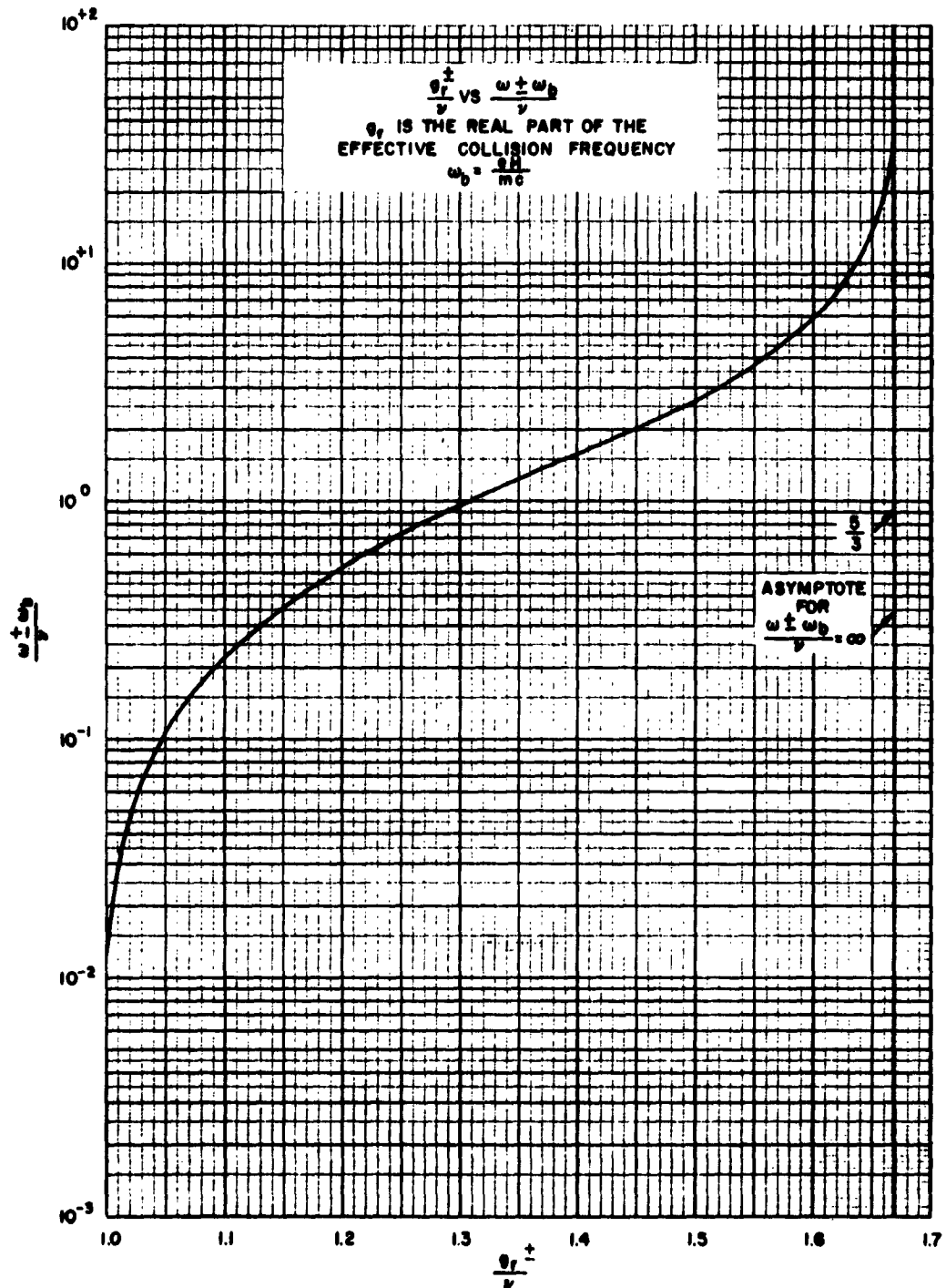


Figure 1

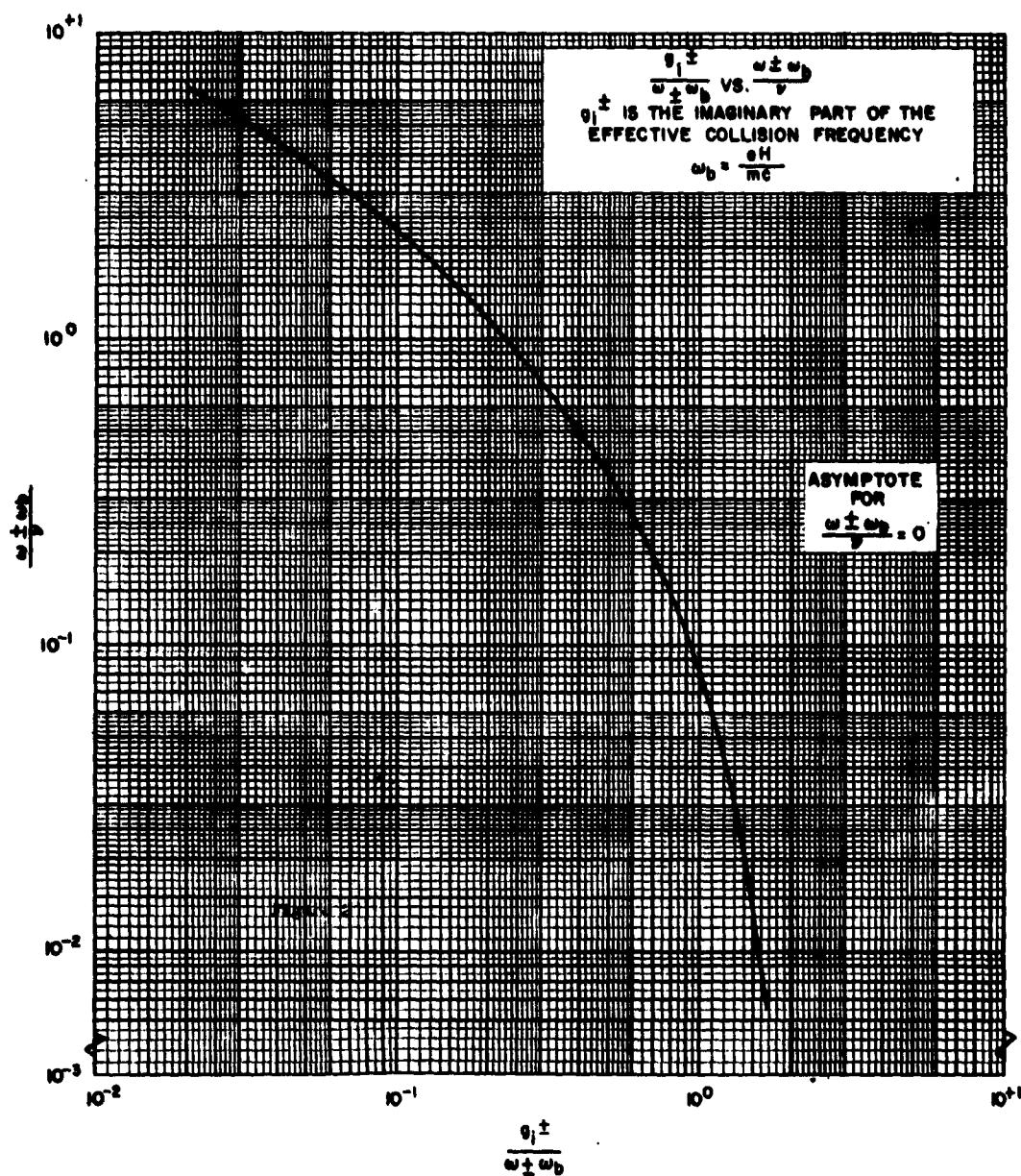


Figure 2